Math-109: Pre-Calculus Algebra
Section: 8
Midterm Exam 1 Solutions

Name: ______________________________
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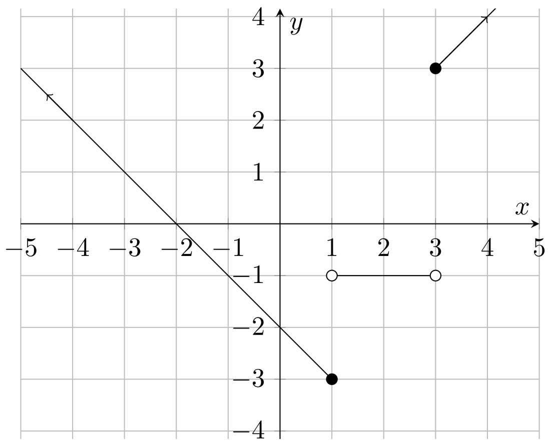
Please write complete step by step solutions to the problems below.

1. Consider the following graph:



(a) Is the graph a function?

Solution. Yes because it satisfies the vertical line test. □

(b) Is this a one-to-one function?

Solution. No because it does not satisfy the horizontal line test. □

(c) Determine the domain of the function. (Note the arrows at the end).

Solution. $(-\infty, \infty)$ □

(d) Determine the range of the function. (Note the arrows at the end).

Solution. $[-3, \infty)$

□

(e) Determine the intervals where the function is increasing, decreasing and constant.

Solution. Increasing on $(3, \infty)$.
Decreasing on $(-\infty, 1)$
Constant on $(1, 3)$ □

(f) Find all x in the domain such that $f(x) = -1$.

Solution. We draw a horizontal line through -1 and notice the points where it intersects the graph. The input values of those points are $\{-1\} \cup (1, 3)$. □

2. Determine the domains of the following functions:

   (a) $f(x) = \frac{x-1}{x-9}$

   Solution. Since the denominator cannot be 0 we need to find the values of x for which the denominator is 0. We know that $x - 9 = 0$ for $x = 9$. Thus, we have to exclude 9 from our domain. Hence, the domain is $(-\infty, 9) \cup (9, \infty)$. □

   (b) $g(x) = \frac{1}{\sqrt{x+2}}$

   Solution. There is a sqaure root and also a denominator.
   Let us first deal with the square root:
   Since the value inside the square root must be non-negative, $x + 2 \geq 0$. Hence, $x \geq -2$ (*).
   Now let us deal with the denominator:
   The denominator cannot be 0. The denominator is 0 when $\sqrt{x+2} = 0$. Squaring both sides we get that $x = -2$. Hence, we must exclude -2 from our domain. (**)
   The numbers that satisfy both condition (*) and (**) are $(-2, \infty)$. □

3. Determine whether the following functions are even, odd or neither. Show work

   (a) $f(x) = 2x^2 + |x| + 1$

   Solution. We have

$$\begin{aligned} f(-x) &= 2(-x)^2 + |-x| + 1 \\ &= 2x^2 + |x| + 1 && \text{since } (-x)^2 = x^2 \text{ and } |-x| = |x| \\ &= f(x) \end{aligned}$$

   Since, $f(-x) = f(x)$, f is even. □

   (b) $g(x) = \frac{1}{x} + 3x$

   Solution. We have

$$\begin{aligned} g(-x) &= \frac{1}{-x} + 3(-x) \\ &= -\frac{1}{x} - 3x \end{aligned}$$

   This is not equal to $g(x)$. Hence, g is not even. Now let us factor out the negative sign from above. Then we get

$$\begin{aligned} g(-x) &= -(\frac{1}{x} + 3x) \\ &= -g(x) \end{aligned}$$

   Since $g(-x) = g(x)$, g is odd. □

4. Let $f(x) = 3x + 1$ and $g(x) = x^2 + 3$. Find the expressions for the following and simplify:

   (a) *Solution.*

$$\begin{aligned} f + g &= 3x + 1 + x^2 + 3 \\ &= x^2 + 3x + 4 \end{aligned}$$

   □

(b) $f - g$

Solution.

$$\begin{aligned} f - g &= 3x + 1 - (x^2 + 3) \\ &= 3x + 1 - x^2 - 3 \\ &= -x^2 + 3x - 2 \end{aligned}$$

□

(c) $\frac{f}{g}$

Solution.

$$\frac{f}{g} = \frac{3x+1}{x^2+3}$$

□

(d) $g(f(1))$

Solution.

$$\begin{aligned} g(f(1)) &= g(3 \cdot 1 + 1) \\ &= g(4) \\ &= 4^2 + 3 \\ &= 19 \end{aligned}$$

□

5. Evaluate the following for the given functions $f(x) = x^3 - 1$, $g(x) = |x| + 1$.
$\frac{f(2)-g(-1)}{g(1)}$

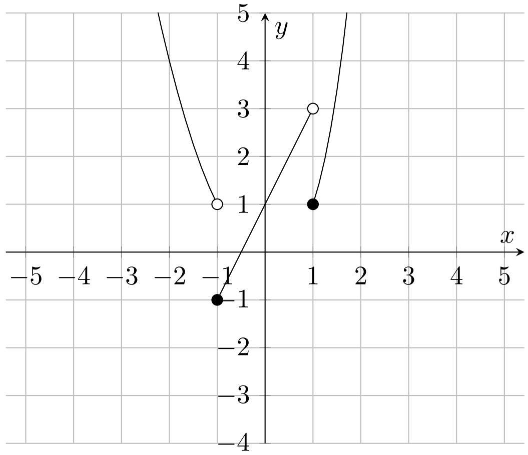
Solution. We have $f(2) = 2^3 - 1 = 7$,
$g(-1) = |-1| + 1 = 1 + 1 = 24$, and
$g(1) = |1| + 1 = 1 + 1 = 2$. Thus,

$$\begin{aligned} \frac{f(2) - g(-1)}{g(1)} &= \frac{7-2}{2} \\ &= \frac{5}{2} \end{aligned}$$

□

6. Graph the piecewise-defined function.

$$f(x) = \begin{cases} x^2, & x < -1, \\ 2x + 1, & -1 \le x < 1, \\ x^3, & 1 \le x. \end{cases}$$



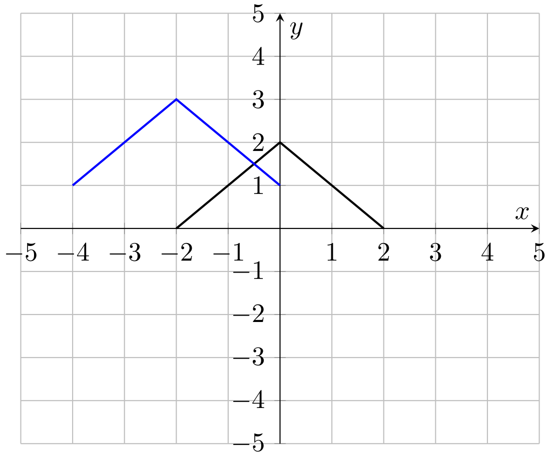
7. Find the difference quotient for the function $f(x) = x^2 + 2x$.

Solution.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h} \\ &= \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\ &= \frac{2xh + h^2 + 2h}{h} \\ &= \frac{h(2x + h + 2)}{h} \\ &= 2x + h + 2 \end{aligned}$$

□

8. Refer to the graph of $y = f(x)$ in the accompanying figure to sketch the graph of $y = f(-x-2)+1$



Solution. We have to becareful here as there is a trap is easy to fall into. The order of the transformation definitely matters. I will perform the following sequence of transformations:
$f(x) \xrightarrow{\text{shift right}} f(x-2) \xrightarrow{\text{shift up}} f(x-2)+1 \xrightarrow{\text{reflect Y axis}} f(-x-2)+1$ It would be incorrect to do the following sequence:
$f(x) \xrightarrow{\text{reflect Y axis}} f(-x) \xrightarrow{\text{shift right}} f(-(x-2)) = f(-x+2)$ but we want $f(-x-2)$.

□

9. Write the expression of the function whose graph is transformed accordingly.

   (a) The graph of $y = \sqrt{x}$ reflected about the x-axis, and then shifted left 4 units.

   Solution. $\sqrt{x} \xrightarrow{\text{reflect x-axis}} -\sqrt{x} \xrightarrow{\text{shift left}} -\sqrt{x+4}$ □

   (b) The graph of x^3 horizontally stretched by a factor of $\frac{1}{2}$, shifted down by 2 units, and then reflected about the x-axis.

   Solution. $x^3 \xrightarrow{\text{stretch horizontally}} (\frac{1}{2}x)^3 \xrightarrow{\text{shift down}} (\frac{1}{2}x)^3 - 2 \xrightarrow{\text{reflect x axis}} -\left[(\frac{1}{2}x)^3 - 2\right] = 2 - (\frac{1}{2}x)^3$ □

10. Given $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{1-x}$, find

   (a) $(f \circ g)(x)$

   Solution.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\frac{1}{1-x}) \\ &= \frac{1}{\frac{1}{1-x}} \\ &= 1 - x\end{aligned}$$

□

   (b) $(g \circ f)(x)$

Solution.

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(\frac{1}{x}) \\
&= \frac{1}{1 - \frac{1}{x}} \\
&= \frac{x}{x-1}
\end{aligned}
$$

□

11. Find the inverse of $f(x) = \frac{1}{3x+5}$.

Solution. Let y be in the range of f. Let x be in the domain such that $f(x) = y$. So $y = \frac{1}{3x+5}$. Given x we know how to find the output y. Now we want to know how to find x given the y. Hence, we have to isolate the x on one side and put the rest of the y terms on the other side. Multiplying both sides by $3x + 5$ we get,

$$
\begin{aligned}
(3x+5)y &= 1 \\
3xy + 5y &= 1 \\
3xy &= 1 - 5y \\
x &= \frac{1-5y}{3y} && \text{Diving both sides by 3y}
\end{aligned}
$$

Therefore, $f^{-1}(y) = \frac{1-5y}{3y}$. □

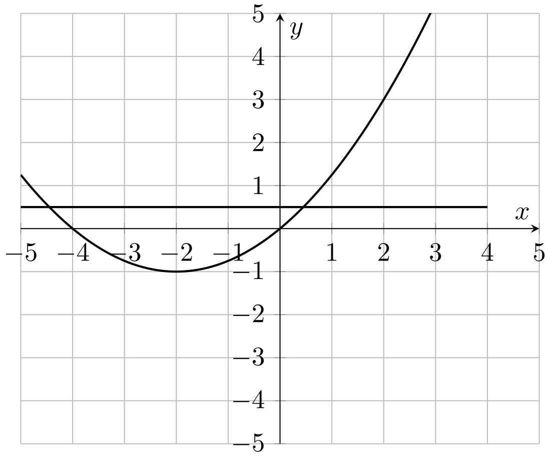
12. Determine whether the function $f(x) = 2(\frac{x}{2}+1)^2 - 1$ is one to one. You can either show it algebraically or graphically. If you choose to do it graphically, then you must show the appropriate step by step transformations of the common function clearly.

Solution. Algebraic method:
Let x_1 and x_2 be in the domain X such that $f(x_1) = f(x_2)$. Hence,

$$
\begin{aligned}
2(\frac{x_1}{2}+1)^2 - 1 &= 2(\frac{x_2}{2}+1)^2 - 1 \\
2(\frac{x_1}{2}+1)^2 &= 2(\frac{x_2}{2}+1)^2 \\
(\frac{x_1}{2}+1)^2 &= (\frac{x_2}{2}+1)^2
\end{aligned}
$$

But we can find two numbers which are not equal but whose squares are equal. For example $2^2 = (-2)^2 = 4$. Let $\frac{x_1}{2} + 1 = -2$ and $\frac{x_2}{2} + 1 = 2$. Then we get that $x_1 = 2$ and $x_2 = -6$. so $f(2) = f(-6)$ but $2 \neq -6$.

Geometric method:



By the horizontal line test, the function is not one-to-one. But you have to show the transformation steps. $x^2 \to ...$

$\square$

BONUS QUESTIONS

1. Use transformations of functions to plot the graph of $f(x) = 2 - \frac{1}{3(1-\frac{x}{2})}$. Write the step by step transformations.

2. You are given 80 coins of the same denomination; you know that one of them is counterfeit and that it is lighter than the others. Locate the counterfeit coin by using four weighings on a pan balance.